THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010I/J University Mathematics 2015-2016
Revision for Midterm Examination

1. Let $\{a_n\}$ be a sequence of real numbers defined by

$$a_1 = \sqrt{2} \qquad \text{and} \qquad a_{n+1} = \sqrt{2 + a_n} \text{ for } n \geq 1.$$

   (a) Show that $\{a_n\}$ is an increasing sequence.

   (b) Show that $\{a_n\}$ is convergent and hence find the $\lim_{n\to\infty} a_n$.

2. Let $\{x_n\}$ be a sequence of real numbers defined by

$$x_1 = 3 \qquad \text{and} \qquad x_{n+1} = \frac{x_n^2 + 4}{2x_n} \text{ for } n \geq 1.$$

   (a) Prove that $0 \leq x_n - 2 \leq \dfrac{1}{2^{n-1}}$ for all natural numbers n.

   (b) Prove that $\{x_n\}$ converges and find $\lim_{n\to\infty} x_n$.

3. Evaluate the following limits.

   (a) $\lim_{x\to+\infty} \dfrac{x}{1 + \sqrt[3]{x^3 + 1}}$

   (b) $\lim_{x\to+\infty} \dfrac{x + \sin x}{x - \cos x}$

   (c) $\lim_{x\to 0} \dfrac{\tan 3x}{4x}$

4. Find the derivative of the following functions.

   (a) $f(x) = e^{\cos 3x}$

   (b) $f(x) = x^{\sin x}$, for $x > 0$

   (c) $f(x) = \sin^{-1}(\sin^2 x)$

   (d) $f(x) = \dfrac{(x^2 + 1)^3}{e^{3x}(\sqrt{x} + 3)^4}$

5. Show that $\lim_{x\to 0} \sin(e^{(1/x^2)})$ does not exist.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

- $f(x+y) = f(x) + f(y) + 2xy(x+y)$ for all real numbers x and y;
- $f'(0) = 1$.

Show that $f'(x) = 1 + 2x^2$ for all real numbers x.

7. Let $a > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{1}{a}(x^2 - a^2) & \text{if} \quad 0 < x < a; \\ 0 & \text{if} \quad x = a; \\ \dfrac{a}{x^2}(x^2 - a^2) & \text{if} \quad x > a. \end{cases}$$

(a) Prove that $f(x)$ is differentiable at $x = a$ and find $f'(a)$.

(b) Is $f'(x)$ continuous at $x = a$?

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

- $f(x+y) = f(x)f(y)$ for all real numbers x and y;
- $1 + x \leq f(x) \leq 1 + xf(x)$ for all real numbers x.

(a) Show that

(i) $f(0) = 1$,

(ii) $f(x) > 1$ for $x > 0$,

(iii) $f(x) > 0$ for all real number x.

Hence, deduce that $f(x)$ is strictly increasing, that means if $a > b$, then $f(a) > f(b)$.

(b) Show that if $h < 1$, we have

$$1 + h \leq f(h) \leq \frac{1}{1-h}.$$

Hence, show that $f(x)$ is continuous at $x = 0$.

(c) Show that $f(x)$ is differentiable at $x = 0$ and find $f'(0)$.